Energy equipartition starting from high-frequency modes in the Fermi-Pasta-Ulam β oscillator chain

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We study the approach to equipartition in the Fermi-Pasta-Ulam oscillator chain with quartic nonlinearity Fermi-Pasta-Ulam–(β system) starting from generic high-frequency-mode initial conditions. Typically 90% of the energy is placed in one high-frequency mode, with 10% in adjacent modes. The mode energy is found to distribute itself into first a number of localized structures which coalesce over time into a single localized structure, a chaotic breather (CB). Over longer times the CB is found to break up, with energy transferred to lower frequency modes which do not have the breather symmetry. A transition with decreasing initial mode frequency is found such that the CB does not form, as expected from the loss of breather symmetry. The scaling of CB formation time with energy density, E/N, is found to be $T_b \propto (E/N)^{-1}$, and the scaling of equipartition time found to be $T_{\mathrm{eq}} \propto (E/N)^{-2}$. The scaling of T_{eq} can be predicted from an argument which postulates stochastic diffusion from high-frequency-mode chaotic beat oscillations to the low-frequency modes. The theory also predicts that a miminum value of E/N exists below which T_{eq} should increase more rapidly with E/N than in the power law range, and this transition has been found numerically.

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I. INTRODUCTION

Coupled oscillator chains form good test systems for investigating energy exchange among degrees of freedom [1]. In particular, the Fermi-Pasta-Ulam (FPU) system, consisting of a set of equal masses coupled to nearest neighbors by nonlinear springs, has been extensively studied [1–13]. Starting with energy initially in a low-frequency mode, Fermi *et al.* [2] observed, for low energies, that the oscillators did not relax to the equipartition state, but displayed recurrences which were later explained in terms of beating among the system modes [1,3]. A theoretical prediction of a threshold to fast equipartition by mode overlap [4] was subsequently qualitatively confirmed by studies of energy thresholds required to give approximate equipartition among modes [5–7]. A weaker mechanism that also led to equipartition on a slower time scale has also been studied [8–10]. With initial energy in a low-frequency mode, it was shown in Ref. [9] that the resonant interaction of a few low-frequency modes can lead to local superperiod beat oscillations that are stochastic, transferring energy to high-frequency modes by diffusion. With increasing local energy, there is a transition from exponentially slow transfer to a time scale that is inversely proportional to a power of the energy density.

It has also been shown that the FPU-β system with quartic nonlinearity can be approximated, for low-frequency-mode initial conditions, by the mKDV equation, which admits a soliton solution, that can become unstable with increasing energy [11]. It was further demonstrated that this instability roughly coincides with the creation of stochastic layers in the beat oscillations [9]. The close connection between the development of stochastic layers in beat oscillations and instabilities in nonlinear structures was also noted for the discretized sine-Gordon equation, consisting of pendula coupled by linear springs [14,15]. In Ref. [14], it was numerically found that the breakup of a nonlinear structure, starting from a high-frequency-mode initial condition, occurred at higher energy (and on a slower time scale) than from energy initially in a low-frequency mode.

A partial understanding of the increased stability came from a series of analyses of breatherlike structures on discrete systems that admitted exact breather solutions [12,16–18,13]. The high-frequency-mode initial conditions have symmetry of neighboring oscillators close to that of the localized exact breathers. The resulting dynamics consists of two stages. First there is an initial period in which the mode breaks up into a number of breatherlike structures which coalesce into one large unstable structure. These structures have been called chaotic breathers (CB) [13]. Since a single large CB closely approximates a stable breather, the final decay stage, toward equipartition, is slow. This behavior has been observed in oscillator chains approximating the Klein-Gordon equation with various force laws [16–18], e.g., the discretized sine-Gordon equation [18], and, more relevantly for this paper, the FPU-β model [12,13]. In these latter works, the energy was placed in the highest-frequency mode with strict alternation of the amplitudes from one oscillator to the next. This configuration is stable up to a particular energy at which a parametric instability occurs, leading to the events described above [12,13]. However, the nonlinear evolution does not depend on such special initial conditions, but will generically evolve from any high-frequency-mode initial condition that has predominantly the alternating amplitude symmetry. One does not know, in this generic situation, whether there exists any true energy threshold to achieve equipartition, but as discussed extensively with respect to low-frequency-mode initial conditions, the practical thresholds refer to observable time scales [9,10]. From a phase space perspective it is intuitively reasonable that for a large number of oscillators and not too low an initial energy a generic set of initial conditions will lie in a chaotic layer,

but the chaotic motion can remain close to a regular orbit for very long times [1].

In this paper we follow the studies of high-frequency modes in Refs. [12,13], but for more generic initial conditions. We numerically study the equipartition time scale, confirming the scaling found in Ref. [13] for our more generic case. Although the central importance of the existence of CB's is confirmed, we also emphasize the dual significance of the modes and the beat phenomenon among modes that has been shown to be of central importance to understanding equipartition with low-frequency initial conditions [9,19]. In particular, we show how these beats can be used to predict the scaling of the time to equipartition as a function of energy density.

II. NUMERICAL CALCULATIONS

The Hamiltonian representing the FPU-β chain of N oscillators is [1,2]

$$H=\sum_{i=1}^{N}\frac{p_i^2}{2}+\sum_{i=0}^{N}\left[\frac{1}{2}(q_{i+1}-q_i)^2+\frac{\beta}{4}(q_{i+1}-q_i)^4\right]. \quad (1)$$

We consider the case of strong springs ($\beta>0$) and fixed boundaries $q_0=q_{N+1}=0$. The constant β describing the strength of the anharmonic potential can be scaled to any positive value. We vary the energy and fix β at the commonly used value 0.1 to compare with previous studies. The equations of motion, obtained from the Hamiltonian (1), are numerically integrated using a fourth order symplectic integrator [20] with a fixed integration step. The harmonic part of the Hamiltonian can also be put in the form of N independent normal modes via the canonical transformation

$$Q_j=\sqrt{\frac{2\Omega_j}{N+1}}\sum_{i=1}^{N}\sin\left(\frac{\pi ij}{N+1}\right)q_i, \quad (2)$$

$$P_j=\sqrt{\frac{2}{\Omega_j(N+1)}}\sum_{i=1}^{N}\sin\left(\frac{\pi ij}{N+1}\right)p_i, \quad (3)$$

where the frequencies Ω_j of the normal modes $\{Q_j,P_j\}$ are

$$\Omega_j=2\sin\left(\frac{\pi j}{2N+2}\right). \quad (4)$$

The main statistical tools we use to examine the evolution of the energy distribution are the normalized effective number of normal modes containing energy,

$$n_{\mathrm{eff}}=\frac{1}{N}\exp\left[-\sum_{j=1}^{N}e_j\ln e_j\right], \quad (5)$$

where $e_j=E_j/\Sigma_{j=1}^{N}E_j$ are the normalized linear energies of the normal modes $[E_j=(1/2)\Omega_j(Q_j^2+P_j^2)]$; and the normalized effective number of oscillators containing energy,

$$n_{\mathrm{osc}}=\frac{1}{N}\exp\left[-\sum_{i=1}^{N}e_i\ln e_i\right], \quad (6)$$

where the e_i are the normalized oscillator energies (in order to distinguish between normal mode energies and oscillator energies we label the former e_j and the latter e_i in this work), which are given by

$$e_i=\frac{1}{N}\left\{\frac{1}{2}p_i^2+\frac{1}{4}[(q_{i+1}-q_i)^2+(q_i-q_{i-1})^2]+\frac{\beta}{8}[(q_{i+1}-q_i)^4+(q_i-q_{i-1})^4]\right\}. \quad (7)$$

Both the quantities n_{eff} and n_{osc} vary in the range from 0 to 1 and have small values for energies concentrated, respectively, in a few modes or oscillators and large values (close to 1) for energies distributed evenly in the modes or oscillators. We can plot these quantities versus time for various initial conditions, values of N, and energy densities E/N. For statistical quantities we also average over several (typically 5–20) different realizations of the initial mode phases to reduce fluctuations.

We first concentrate our attention on a particular value of $N=128$, which is sufficiently large to avoid small N effects, but small enough for faster calculations. We choose a generic high-frequency-mode initial condition with the mode $\gamma=120$ containing 90% of the energy and the remaining 10% in the adjacent modes. More specifically, we choose a total system energy of $E=50$. The primary oscillator symmetry in this case is alternating amplitudes. As in cases previously studied [12,13], in which the energy is placed in the highest mode, the mode energy rapidly breaks up into a number of traveling oscillator-localized nonlinear structures, which then coalesce over time into a single dominant localized structure; these are the chaotic breathers (CB). Unlike the results for the initial conditions used in previous studies, there is no abrupt onset of the CB formation with increasing energy. Rather the time of formation lengthens as the energy is decreased, and probably becomes exponentially long at sufficiently low energy. Following Ref. [13] we trace the evolution of localized structures in Fig. 1, with the gray scale quantitatively corresponding to the energy in the localized oscillator sites. We concentrate on three times: a time at which there are several CB's being formed, a somewhat later time with only one CB and a time when the dominant CB has substantially decayed. Corresponding to the three times we show the oscillator energies in Fig. 2, and the mode energies in Fig. 3. The sharp breather structures are noted in Figs. 2(a) and 2(b), and the envelope $\sin(x)/x$ pattern of modes (averaged in time), centered on $j=128$, noted in Figs. 3(a) and 3(b). In Fig. 2(c) the CB has mostly dissipated, while in Fig. 3(c) the energy has spread substantially (but still well short of equipartition) to low-frequency modes that do not have the breather symmetry. To emphasize the breather symmetry we show, in Fig. 4, the mode amplitudes of the principal modes in the breather over a short time corresponding roughly to Figs. 2(b) and 3(b). Over the short observation time the breather is concentrated primarily on oscillators 9 and 10. The alternating symmetry is evident, and the amplitudes oscillate at the breather frequency $\omega_b\approx 2.6(\tau_b\approx 2.4\,\mathrm{s})$. Finally, in Figs. 5 and 6 we give the statistical measures n_{osc} and n_{eff} as functions of t, including again the times in Figs. 2 and 3. We note that n_{osc} illustrates

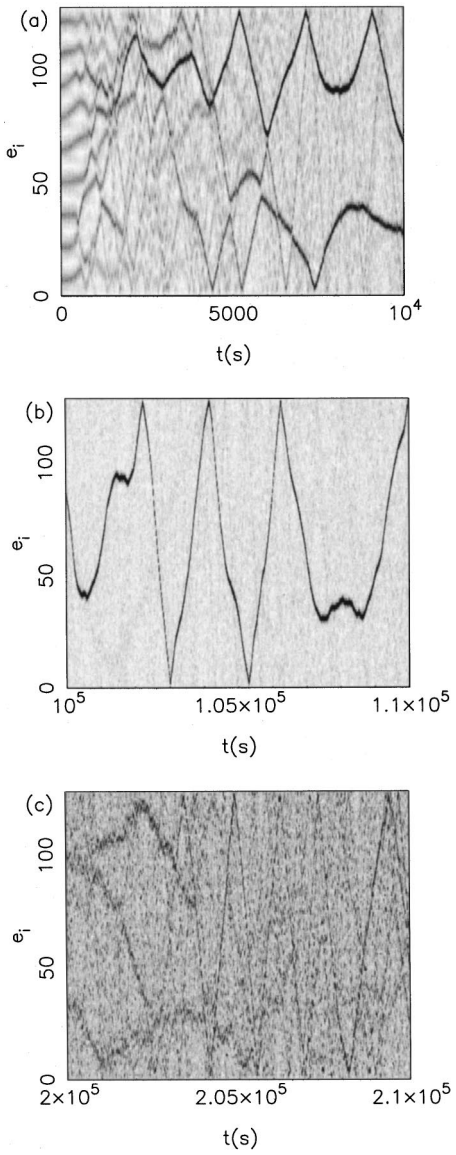

FIG. 1. Time evolution of the energy distribution in the oscillators of the FPU-β oscillator chain with $\beta=0.1$, $E=50$, $N=128$ oscillators and initial energy concentrated around the mode $\gamma=120$, at times (a) $0<t<10^4$ s, (b) $10^5<t<1.1\times10^5$, and (c) $2\times10^5<t<2.1\times10^5$. Darker regions correspond to oscillators with more energy, lighter regions to oscillators with less energy.

the formation and decay of the breathers. In contrast, n_{eff}, after an initial transient in which the multiple breathers form, hides the dynamics in a slow statistical progression towards equipartition.

Another question to be considered is the effect of initial conditions. Taking the energy as in Figs. 1–6 and the time such that the breather has formed at this energy, we examine various initial conditions in Fig. 7. The breather symmetry, which is exact for all energy initially in the mode $\gamma=128$, should be completely lost for $\gamma=64$ and below. This loss of breather symmetry is quite evident in Fig. 7, which indicates that the breather will essentially pick up its full energy for all phases above (approximately) $\gamma=105$, below which deterioration of the breather occurs. After a rather rapid transition the breather has been essentially eliminated by $\gamma=85$.

We now consider the effect of varying parameters. In Fig. 8 we illustrate for various values of E at fixed $N=128$ and $\gamma=120$ the times required for breather formation, for the breather to approximately maintain its shape, and for the breather to disintegrate. As in Fig. 5, n_{osc} is used to distinguish the regimes. We observe that all times decrease with increasing E. The maintenance of the CB and its deterioration cannot be easily separated, and may be collectively designated as the time to reach equipartition.

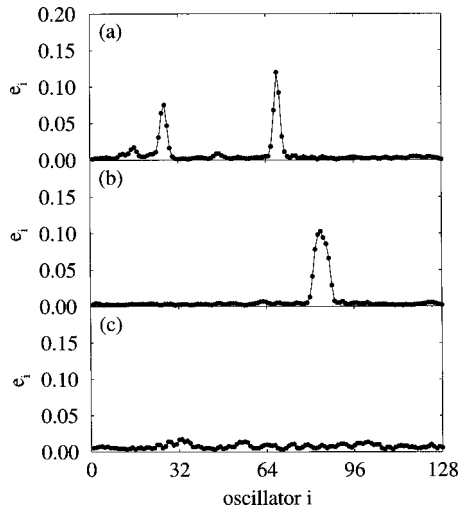

FIG. 2. Energy density distributions among the oscillators for a FPU-β system with $\beta=0.1$, $E=50, N=128$ oscillators and $\gamma=120$ at times (a) $t=1\times10^4$ s, (b) $t=1\times10^5$ s, and (c) $t=2\times10^5$ s.

In Fig. 9 we plot $\ln T_{\mathrm{eq}}$ vs $\ln N$, where T_{eq} is the time for n_{osc} to increase to $n_{\mathrm{osc}}=0.7$. We see, for fixed $\gamma/N=15/16$ and fixed $E/N=0.625$, that there is a transition, approximately at $N=128$ where T_{eq} changes from a decreasing function of N to a constant, independent of N. For large N, T_{eq} has also been found to be independent of N with low-frequency-mode initial conditions. However, this scaling can be obscured by early-in-time transients for some parameters. The main scaling is with energy density, which we give in

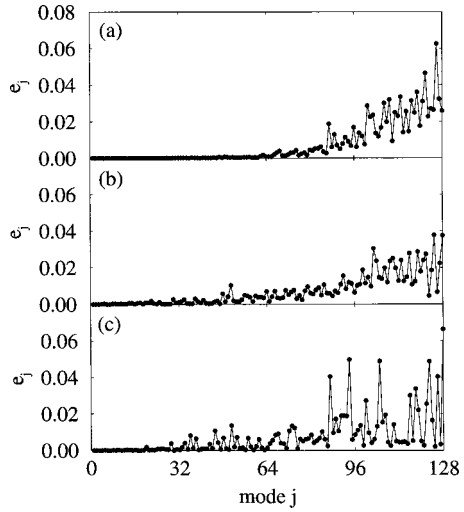

FIG. 3. Energy density distributions in the mode space for a FPU-β system with $\beta=0.1$, $E=50$, $N=128$ oscillators and $\gamma=120$ at times (a) $t=1\times10^4$ s, (b) $t=1\times10^5$ s, and (c) $t=2\times10^5$ s.

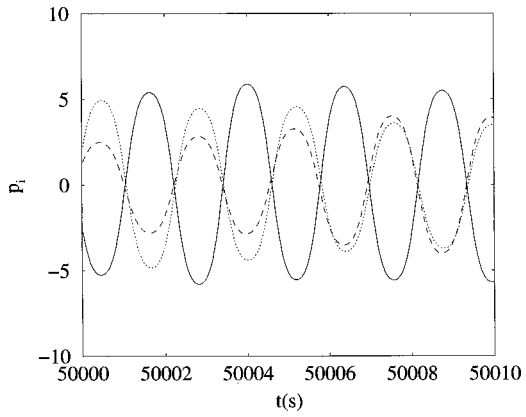

FIG. 4. Breather phase oscillations for the oscillators at sites $i=9$ (dotted line), $i=10$ (solid line), and $i=11$ (dashed line) for $\beta=0.1$, $E=50$, $\gamma=120$, and $N=128$ oscillators. The oscillator $i=10$ has frequency $\Omega_i=2.62\,\mathrm{s}^{-1}$.

Fig. 10. Plotting $\ln T_{\mathrm{eq}}$ vs $\ln(E/N)$ we find a slope well approximated by -2, i.e., $T_{\mathrm{eq}} \propto (E/N)^{-2}$. In the next section we present an argument from which this scaling can be estimated. In Fig. 11 we give the scaling with energy density of the time for CB formation T_b, defined as the time for n_{osc} to fall to $n_{\mathrm{osc}}=0.3$. For a single breather to form we require that $T_b \lesssim T_{\mathrm{eq}}$. We find this inequality to hold over the range of E/N values we have investigated, which corresponds to cases in which CB's form. However, we note that the scaling of the breather formation time is approximately $T_b \propto (E/N)^{-1}$. Thus extrapolations of T_{eq} and T_b to higher energy predict a crossing, which is a transition to energy densities beyond which single breathers will not form. Of course the transition is qualitative, in the sense that the definitions of the times for formation and destruction of the CB are also qualitative.

III. ESTIMATING THE TIME TO EQUIPARTITION

In the usual picture of breather stability, the physical mechanism by which the breather loses stability is that the breather frequency becomes resonant with a linear normal mode [16–18]. This explanation is not directly applicable to our problem as the breather frequency is higher than the highest normal mode; e.g., in Fig. 4 the CB has a frequency $\omega_b=2.62$ while the highest mode frequency is $\max(\Omega_j)\approx 2$. However, we know this breather is unstable (a CB), as it must have been formed in the chaotic portion of the Hamiltonian phase space in order that is was able to be formed from a few initial modes. Within the usual theory the process then becomes quite subtle, as it depends on the relatively small continuous spectrum of the chaos.

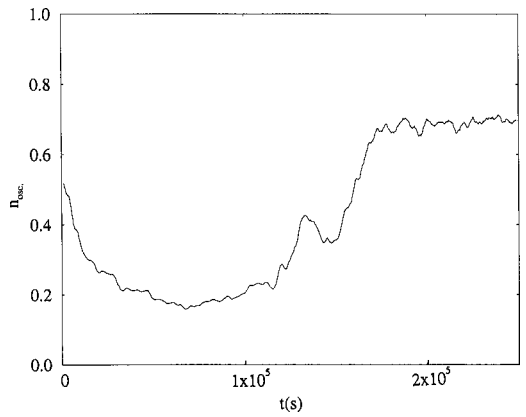

FIG. 5. Time evolution of the normalized effective number of oscillators (n_{osc}) for $\beta=0.1$, $E=50$, $\gamma=120$, and $N=128$ oscillators.

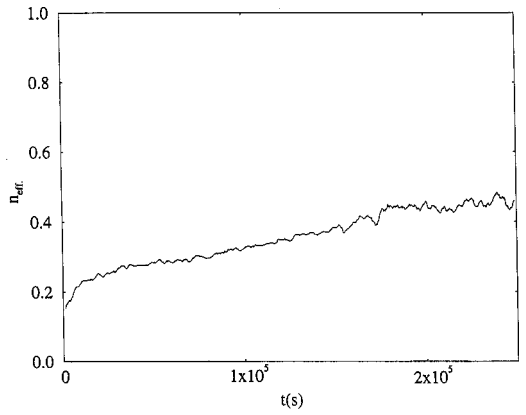

FIG. 6. Time evolution of the normalized effective number of modes (n_{eff}) for $\beta=0.1$, $E=50$, $\gamma=120$, and $N=128$ oscillators.

Although the dominant structure is the CB, the mode spectrum, into which the CB can be decomposed, plays an important role. In particular, adjacent modes interact to form beats on a slower time scale, which can then interact by the Arnold diffusion mechanism to drive energy to low-frequency modes which lack the breather symmetry. This transfer of energy from the high-frequency portion of the spectrum to the low-frequency portion is probably the dominant energy transfer mechanism; it has been shown to be the dominant energy transfer mechanism from low-frequency-mode initial conditions to the high-frequency modes [9]. In a recent work [19] we have found that the scaling with energy density of the time to reach equipartition can be predicted from that mechanism. The proportionality $T_{\mathrm{eq}}(\text{low-to-high}) \propto (E/N)^{-3}$ was predicted and confirmed numerically. Here we show that the same formalism can predict the scaling $T_{\mathrm{eq}}(\text{high-to-low}) \propto (E/N)^{-2}$, which we have found numerically in Fig. 10. We only quote some results of the theory.

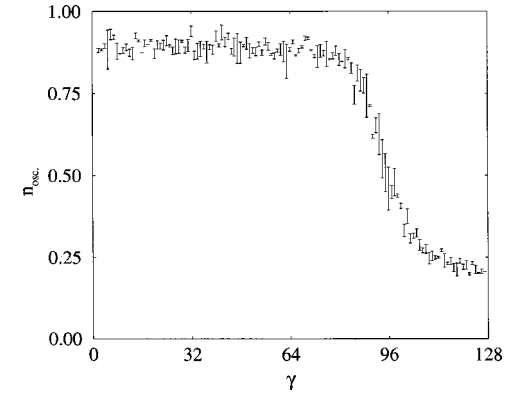

FIG. 7. Averages of the normalized effective number of oscillators (n_{osc}) over eight different initial conditions vs the initial energy mostly in mode γ for $\beta=0.1$, $E=50$, $N=128$ oscillators and $t=5\times10^4$ s.

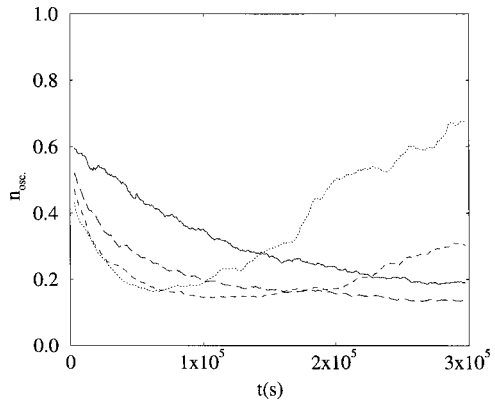

FIG. 8. Averages of the normalized effective number of oscillators ($n_{\rm osc}$) over eight different initial conditions vs time for $\beta=0.1$, $\gamma=120$, $N=128$ oscillators and $E=14$ (solid line), 26 (long-dashed line), 38 (dashed line), and 50 (dotted line).

The reader is referred to the original papers for a more detailed treatment [9,19]. Before outlining the steps, we show relevant beat oscillations in Fig. 12. The energies in two adjacent modes (other modes are also involved) are shown, with a small fast energy interchange associated with the CB and a large slow energy interchange at the main beat frequency. The scaling of the main beat frequency with energy density is found, numerically, to be $\Omega_B \propto (E/N)^{0.8}$.

To theoretically estimate the scaling of the time to equipartition we transform the Hamiltonian (1) to action-angle variables of the normal modes. We first transform to the normal mode variables, using (2) and (3), and then introducing the canonical action-angle variables (I, ϕ) through the transformations $Q_i=\sqrt{(2I_i/\Omega_i)}\cos(\phi_i)$ and $P_i=\sqrt{(2\Omega_i I_i)}\sin(\phi_i)$ we obtain

$$H=\sum_j \Omega_j I_j+\left(\frac{\beta}{8N+8}\right)\sum_{i,j,k,l} G(i,j,k,l) \times\sqrt{\Omega_i\Omega_j\Omega_k\Omega_l I_i I_j I_k I_l}\,ang(ijkl), \tag{8}$$

where $ang(ijkl)\equiv\cos(\phi_i)\cos(\phi_j)\cos(\phi_k)\cos(\phi_l)$. The coefficients G, as calculated in Refs. [3,9] are

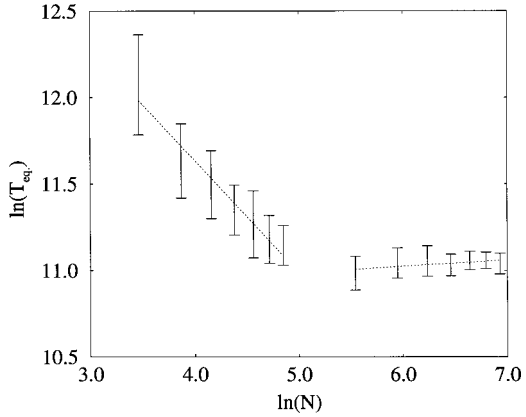

FIG. 9. Natural logarithm of time until $n_{\rm osc}>0.7(T_{\rm eq})$ vs natural logarithm of the total number of oscillators for $\beta=0.1$, $E/N=0.625$ and $\gamma/N=15/16$ (each point is obtained by averaging over 20 different initial conditions).

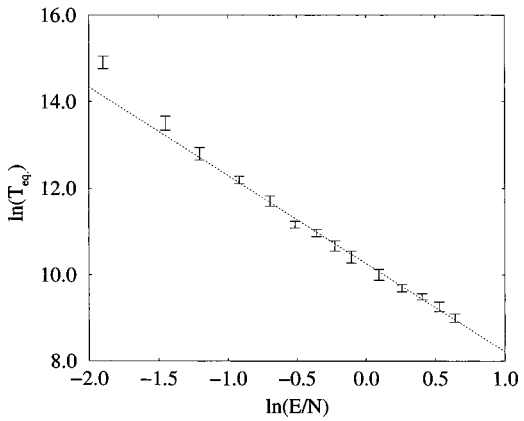

FIG. 10. Natural logarithm of time until $n_{\rm osc}>0.7$ ($T_{\rm eq}$) vs natural logarithm of E/N for $\beta=0.1$, $\gamma=120$, and $N=128$ oscillators (each point is obtained by averaging over eight different initial conditions). The linear fit, obtained over all points but the leftmost two, has a slope of $\alpha=-2.04\pm0.08$.

$$G(i,j,k,l)=\sum_P B(i+j+k+l), \tag{9}$$

where P represents the eight permutations of sign of j, k, and l and the function $B(x)$ takes the value 1 if the argument is zero, -1 if the argument is $\pm2(N+1)$, and zero otherwise. The selection rule (9) follows from the quartic nature of the coupling. We can estimate Ω_B, by taking a derivative of H with respect to an action I_i and then evaluating the nonlinear term in Eq. (8). The derivative reduces the sum by one index, and the selection rule (9) by a second index. If we further consider the sum to run over some δk modes, to be determined, the number of terms in the above sum is then of the order of (δk) [2]. We assume every quartic term in this sum is typically of the same size, i.e., with equal energies for all low-frequency modes, $\Omega_j I_j=E/\delta k$. If we also take the phases to be random, then the effective number of terms is δk, giving the estimate

$$\Omega_B\approx\Omega_j\frac{\beta E}{N}. \tag{10}$$

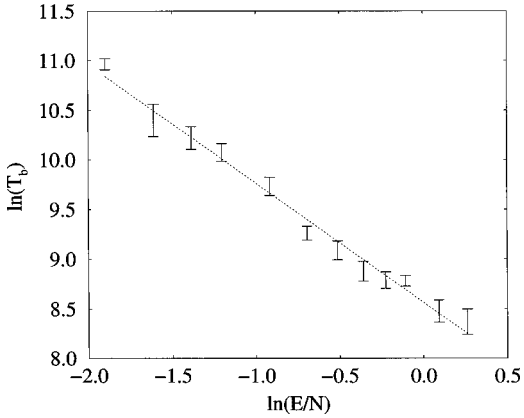

FIG. 11. Natural logarithm of time until $n_{\rm osc}<0.3$ (T_b) vs natural logarithm of E/N for $\beta=0.1$, $\gamma=120$, and $N=128$ oscillators (each point is obtained by averaging over eight different initial conditions). The linear fit has a slope of $\alpha=-1.19\pm0.12$.

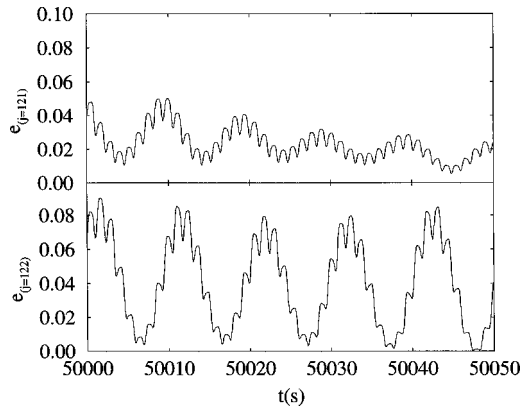

FIG. 12. Time evolution of the energy densities of the modes $j=121$ and $j=122$ for $\beta=0.1$, $E=50$, $\gamma=120$, and $N=128$ oscillators. Two different beat oscillations between the modes can be observed, with respective beat frequencies equal to $\Omega_B^{(1)}\approx 0.60$ and $\Omega_B^{(2)}\approx 5.3$.

For low-frequency modes $\Omega_{j(\mathrm{low})}=\pi j/N$, but for high-frequency modes we can approximate $\Omega_{j(\mathrm{high})}=2$. Note that the scaling of Ω_B with E/N is approximately that found numerically. The key assumption in the calculation is to require, for fast Arnold diffusion [9], that

$$\Omega_B \gtrsim \delta\Omega_l, \tag{11}$$

where $\delta\Omega_l$ is the spread of mode frequencies to which energy can be transferred. For transfer to low-frequency modes

$$\delta\Omega_l=\frac{\pi\delta l}{N}, \tag{12}$$

where δl is the number of low-frequency modes which are taken to correspond one-to-one with high-frequency modes, $\delta l=\delta k$. Using this and combining Eqs. (10)–(12) we obtain the fractional number of couplings

$$\frac{\delta k}{N}=\frac{2}{\pi}\frac{\beta E}{N}. \tag{13}$$

We estimate the rate of energy transfer from a high-frequency mode to the low-frequency modes by taking the derivative of H with respect to angle. Using the same procedure as above, we obtain

$$\frac{dE_j}{dt}\approx\Omega_j\left(\frac{\beta}{N}\right)\delta k E_j E_l, \tag{14}$$

where $\Omega_j=2$, as above, and $\delta k/N$ is given by Eq. (13). With these substitutions we rearrange and integrate to obtain, at equipartition, where $E_j=E/N$,

$$\ln\left(\frac{\pi/2}{\beta E/N}\right)=2\beta\left(\frac{2}{\pi}\frac{\beta E}{N}\right)\int_0^{T_{\mathrm{eq}}}E_l(t)\,dt. \tag{15}$$

As in previous work, we make the simplest assumption that $E_l(t)\approx(t/T)(E/N)$ to perform the integration, yielding

$$T_{\mathrm{eq}}=\frac{\pi}{2}\left(\frac{N}{\beta E}\right)^2\ln\left(\frac{\pi/2}{\beta E/N}\right), \tag{16}$$

which has the dominant $(\beta E/N)^{-2}$ scaling as found numerically. The many approximations are not critical to the scaling, but, of course, change the coefficient, which is certainly no better than an order of magnitude calculation. In fact, if we compare the magnitude of $T_{\mathrm{eq.}}$ from Eq. (16) with the numerical value from Fig. 10, at $E=50$, we find that $T_{\mathrm{eq.}}$ from Eq. (16) is a factor of approximately 50 smaller. We might expect a significant underestimate of T_{eq} because we are not explicitly taking into account the effect of the CB. In fact, for half of the total equipartition time in Fig. 5 the dominant nonlinear processes are holding the CB together. For low-frequency-mode initial conditions, in which there is no breather formation, the theoretical and numerical times are well within an order of magnitude [19]. The above considerations predict that at low initial energy there will be no modes with sufficient energy to produce energy transfer at rates that are nonexponentially slow. From Eq. (13) we find that with $N=128$ and $E=50$, $\delta k\approx 3$ such that reducing the energy by greater than a factor of 3 should result in large increases in T_{eq}. We can see this happening in Fig. 10, where already at $E=26$, T_{eq} is departing from the $T_{\mathrm{eq}}\propto E^{-2}$ scaling.

IV. CONCLUSIONS AND FINAL REMARKS

We have shown that the formation of chaotic breathers (CB's), from high-frequency mode initial conditions in the FPU-β coupled oscillator system, is generic, not dependent on the specific initial conditions. However, the CB formation is successively weakened as the initial conditions contain less of the alternating oscillator breather symmetry. The basic process is the formation of a few CB's from the modes initially containing most of the energy, followed by a coalescence into a single dominant CB, which then decays over time by energy transfer to low-frequency modes which do not contain the breather symmetry. We have found that the dominant scalings are $T_b\propto(E/N)^{-1}$ for CB formation, and $T_{\mathrm{eq}}\propto(E/N)^{-2}$ for CB decay toward equipartition, where E/N is the energy density. Because of the different scalings, at high E/N, $T_b>T_{\mathrm{eq}}$ and the CB will not form.

We have observed, numerically, that a Fourier decomposition into modes shows that there are beat oscillations among the modes, similar to those found for low-frequency-mode initial conditions. The transfer of energy from low to high frequencies was shown to be due to stochasticity, arising in the low-frequency beats, driving energy transfer, nonlinearly, to difference frequencies between high-frequency modes [9]. Postulating that this mechanism also operates to transfer energy from high frequencies to low frequencies, we have shown that it can explain the energy density scaling of $T_{\mathrm{eq}}\propto(E/N)^{-2}$. It further predicts the breakdown of the scaling at low energy density, as found numerically.

The analysis also suggests how the formation of a CB inhibits the energy equipartition process, particularly at lower energies. The concentration of the energy in oscillator space naturally spreads the energy in mode space. If the energy per mode falls significantly below that required for the strong Arnold diffusion, then the transfer of energy from high-frequency to low-frequency modes becomes exponentially small as $\exp(-\delta\Omega_l/\Omega_B)$. Fluctuations in mode amplitude probably assist in the energy transfer process.

There does not presently exist a theory which predicts the formation time of the CB. There are some similarities between the coalescence of CB's and the coalescence of other nonlinear structures, such as vortices. The open questions concerning this intermediate time scale are well worth further investigation.

The additional understanding that has recently been gained for CB formation and destruction from high-frequency-mode initial conditions, combined with the previous understanding of the approach to equipartition from low-frequency-mode initial conditions, now presents a fairly complete picture of the energy transfer phenomenon. The results can probably be applied to other coupled oscillator chains, such as that arising in the approach to equipartition from high and low-frequency modes in the discretized sine-Gordon equation, which previously have been only partially understood.

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